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Lax pair and Darboux transformation for multi-component modified Korteweg–de Vries equations

Hai-Qiang Zhang[1], **Bo Tian**[1,2,3], **Tao Xu**[1], **He Li**[1,4], **Cheng Zhang**[1] and **Huan Zhang**[1]

[1] School of Science, PO Box 122, Beijing University of Posts and Telecommunications, Beijing 100876, People's Republic of China
[2] State Key Laboratory of Software Development Environment, Beijing University of Aeronautics and Astronautics, Beijing 100083, People's Republic of China
[3] Key Laboratory of Optical Communication and Lightwave Technologies, Ministry of Education, Beijing University of Posts and Telecommunications, Beijing 100876, People's Republic of China
[4] Ministry-of-Education Key Laboratory of Fluid Mechanics and National Laboratory for Computational Fluid Dynamics, Beijing University of Aeronautics and Astronautics, Beijing 100083, People's Republic of China

E-mail: gaoyt@public.bta.net.cn (Bo Tian)

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Abstract
In this paper, through generalizing the 2×2 matrix Ablowitz–Kaup–Newell–Segur linear eigenvalue problem to the $2^N \times 2^N$ case, a new Lax pair associated with the multi-component modified Korteweg–de Vries equations is derived in the form of block matrices. Furthermore, the Darboux transformation is applied to this integrable multi-component system, and the n-times iterative potential formula is presented by applying the Darboux transformation successively. This formula enables us to construct a series of explicit solutions of multi-component modified Korteweg–de Vries equations. In illustration, starting from the zero background, we construct the multi-soliton solutions by performing the symbolic computation.

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1. Introduction

The soliton theory and nonlinear evolution equations (NLEEs) are of current importance in many fields of science and technology [1, 2], and hence there has been considerable interest in investigating the complete integrability of the NLEEs [3]. Integrable evolution equations are known to possess many remarkable properties such as the soliton solutions, an infinite number of conservation laws, symmetries and Hamiltonian structures [3–5], and they can be expressed as the compatibility conditions of linear eigenvalue equations (Lax pair) which usually include the spatial part and temporal part [4]. Since the inverse scattering transform (IST) was developed to solve the initial value problem for the Korteweg–de Vries (KdV) equation [6], along with the successful application to the nonlinear Schrödinger (NLS) equation [7] and modified KdV (mKdV) equation [8],

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \tag{2}$$

$$u_t + 6u^2 u_x + u_{xxx} = 0, \tag{3}$$

the Lax pair has been playing an important role in determining whether a given NLEE is integrable and obtaining the explicit solutions of associated integrable equation. Based on the IST method, the Ablowitz–Kaup–Newell–Segur (AKNS) system provides a general scheme to construct the Lax pairs for a large class of physically interesting NLEEs and solve their initial value problems in an extremely systematic way [9].

Many one-component NLEEs can be extended to the versions of multi-component NLEEs. For example, the above KdV, NLS and mKdV equations can be generalized to the following multi-component cases:

$$u_{j_t} + 6\left(\sum_{k=1}^{N} u_k\right) u_{j_x} + u_{j_{xxx}} = 0, \qquad (j = 1, 2, \ldots, N), \tag{4}$$

$$\mathrm{i}q_{j_t} + q_{j_{xx}} + 2\left(\sum_{k=1}^{N} |q_k|^2\right) q_j = 0, \qquad (j = 1, 2, \ldots, N), \tag{5}$$

$$u_{j_t} + 6\left(\sum_{k,l=1}^{N} C_{kl} u_k u_l\right) u_{j_x} + u_{j_{xxx}} = 0, \qquad (j = 1, 2, \ldots, N). \tag{6}$$

The vector generalizations of soliton equations have a significant impact on both theory and phenomenology in nonlinear sciences [10–14]. Mathematically speaking, the multi-component equations admit Painlevé property, Hamiltonian structures, infinite conserved quantities and soliton solutions [15–17]. Moreover, the Lax pairs associated with the multi-component equations are very closely related to the reductions of algebraic group structures [15, 17]. Important physical applications of multi-component equations have received a great deal of attention due to their appearance as models in various areas of physics ranging from fluid mechanics, nonlinear optics to Bose–Einstein condensates and field theories [18–22]. In addition, multi-component equations possess abundant solution structures and appealing soliton collision phenomena [18–20]. In recent studies, the Lax pairs of some multi-component equations can be derived by extending the 2×2 AKNS formulation to the $N \times N$ case [22, 23]. From the viewpoint of algebraic properties, [15, 17] have investigated systems (4)–(6) in (1+1) dimensions associated with Hermitian symmetric spaces. It turns out that system (6) has an infinite number of conservation laws and Hamiltonian structures, and can be solved by the IST method [23, 24].

The Darboux transformation method has been a very effective tool in soliton theory to construct the exact analytical solutions of integrable NLEEs [25–29]. The main idea of this method is to keep the linear eigenvalue problems associated with integrable systems invariant after the appropriate gauge transformation, so that the relationships between the new and original eigenfunctions and potentials can be built. Then, the new solutions of integrable equations can be obtained by solving the linear equations with a trivial solution. Moreover, the advantage of Darboux transformation lies in not only that the solutions are expressible in terms of the Wronskian determinant [25] or Vandermonde-like determinant [30], but that the iterative algorithm is purely algebraic and can be implemented on the symbolic computation system. The effectiveness of this approach has been demonstrated in many integrable one-component NLEEs like the KdV and NLS equations [25, 26].

In recent years, researchers have devoted their attention to the application of the Darboux transformation to integrable multi-component systems regarding the 3-waves resonant interaction equations [31], and coupled NLS equations [20, 32, 33]. With the use of the explicit solutions obtained from the Darboux transformation, the nonlinear phenomena occurring in various fields of physical and engineering sciences can be well illustrated. For example, the resonant interaction of nonlinear waves, elastic and inelastic collisions between solitons and boomeronic phenomenology have been recently found and investigated in [20, 31]. However, when the Darboux transformation is applied to the Lax pair associated with the multi-component systems, the serious problem encountered is how to make the reduction and constraints among original potentials invariable by constructing the appropriate gauge transformation. As far as we know, although there is not a systematic effective method to deal with this problem, the Darboux transformation of some integrable multi-component systems can be constructed with the help of some techniques [20, 31–33].

The purpose of the present work is devoted to making a further investigation on the integrability aspects and multi-soliton solutions of multi-component mKdV equations, i.e., system (6). Through extending the 2×2 matrix AKNS linear eigenvalue problem to the $2^N \times 2^N$ case, we will derive a new Lax pair associated with system (6) in the form of block matrices. Furthermore, based on the obtained Lax pair, we will apply the Darboux transformation method to system (6), and generate the multi-soliton solutions with the purely algebraic iterative algorithm.

2. Lax pairs and reductions

In [23], it has been pointed that system (6) can be transformed into the following normalized multi-component mKdV equations:

$$u_{j_t} + 6\left(\sum_{k=1}^{N} \varepsilon_k u_k^2\right) u_{j_x} + u_{j_{xxx}} = 0, \qquad \varepsilon_k = \pm 1, \quad (j = 1, 2, \ldots, N). \tag{7}$$

In this section, based on the matrix-form inverse scattering formulation [23], we consider the $2^N \times 2^N$ linear eigenvalue problem associated with system (7)

$$\Psi_x = U\Psi = (\lambda J + P)\Psi, \tag{8}$$

$$\Psi_t = V\Psi = (\lambda^3 V_0 + \lambda^2 V_1 + \lambda V_2 + V_3)\Psi, \tag{9}$$

where $\Psi = (\psi_1, \psi_2, \ldots, \psi_{2^N})^T$ (the superscript T denotes the vector transpose) is the vector eigenfunction, λ is the spectral parameter independent of x and t, the block matrices J, P, V_0, V_1, V_2, V_3 are given by

$$J = \mathrm{i}\begin{pmatrix} -I & O \\ O & I \end{pmatrix}, \qquad P = \begin{pmatrix} O & Q \\ R & O \end{pmatrix}, \qquad V_0 = 4\mathrm{i}\begin{pmatrix} -I & O \\ O & I \end{pmatrix}, \qquad V_1 = 4\begin{pmatrix} O & Q \\ R & O \end{pmatrix}, \tag{10}$$

$$V_2 = 2\mathrm{i}\begin{pmatrix} -QR & Q_x \\ R_x & RQ \end{pmatrix}, \qquad V_3 = \begin{pmatrix} Q_x R - Q R_x & -Q_{xx} + 2QRQ \\ -R_{xx} + 2RQR & R_x Q - R Q_x \end{pmatrix}, \tag{11}$$

where I is the $2^{N-1} \times 2^{N-1}$ identity matrix, Q and R are $2^{N-1} \times 2^{N-1}$ block matrices, and O is a $2^{N-1} \times 2^{N-1}$ zero matrix. From the compatibility condition of equations (8) and (9), i.e., the zero-curvature equation

$$U_t - V_x + [U, V] = O, \tag{12}$$

where the brackets denote the commutator of two matrices, the coupled matrix mKdV equations can be obtained

$$Q_t - 3Q_x RQ - 3QRQ_x + Q_{xxx} = O, \tag{13}$$

$$R_t - 3R_x QR - 3RQR_x + R_{xxx} = O. \tag{14}$$

Under the reduction

$$R = \varepsilon Q^T, \qquad \varepsilon = \pm 1, \tag{15}$$

Equations (13) and (14) reduce to the matrix mKdV equation,

$$Q_t - 3\varepsilon(Q_x Q^T Q + QQ^T Q_x) + Q_{xxx} = O. \tag{16}$$

The multi-component mKdV equations (7) can be derived from this single matrix mKdV equation (16).

For a particular case $Q \equiv u$, equation (16) reduces to the standard mKdV equation (3).

If Q is respectively taken as the following forms

$$Q_2 = \left(\begin{array}{c|c} u_1 & u_2 \\ -u_2 & u_1 \end{array}\right)_{2\times 2}, \tag{17}$$

$$Q_3 = \left(\begin{array}{cc|cc} u_1 & u_2 & u_3 & 0 \\ -u_2 & u_1 & 0 & u_3 \\ \hline -u_3 & 0 & u_1 & -u_2 \\ 0 & -u_3 & u_2 & u_1 \end{array}\right)_{4\times 4}, \tag{18}$$

$$Q_4 = \left(\begin{array}{cccc|cccc} u_1 & u_2 & 0 & 0 & u_3 & 0 & u_4 & 0 \\ -u_2 & u_1 & 0 & 0 & 0 & u_3 & 0 & u_4 \\ 0 & 0 & u_1 & u_2 & -u_4 & 0 & u_3 & 0 \\ 0 & 0 & -u_2 & u_1 & 0 & -u_4 & 0 & u_3 \\ \hline -u_3 & 0 & u_4 & 0 & u_1 & -u_2 & 0 & 0 \\ 0 & -u_3 & 0 & u_4 & u_2 & u_1 & 0 & 0 \\ -u_4 & 0 & -u_3 & 0 & 0 & 0 & u_1 & -u_2 \\ 0 & -u_4 & 0 & -u_3 & 0 & 0 & u_2 & u_1 \end{array}\right)_{8\times 8}. \tag{19}$$

It is a direct calculation to verify that the two-component, three-component and four-component mKdV equations of system (7) are obtained with the substitution of expressions (17)–(19) into equation (16), respectively.

The general expression of Q_N is a $2^{N-1} \times 2^{N-1}$ block matrix

$$Q_N = \left(\begin{array}{c|c} \mathbb{Q}_1 & \mathbb{Q}_2 \\ \hline \mathbb{Q}_3 & \mathbb{Q}_4 \end{array}\right), \tag{20}$$

where $\mathbb{Q}_j (j = 1, 2, 3, 4)$ are all $2^{N-2} \times 2^{N-2}$ square-block matrices, $\mathbb{Q}_1$ is a block diagonal matrix, while $\mathbb{Q}_3 = -\mathbb{Q}_2^T, \mathbb{Q}_4 = \mathbb{Q}_1^T$. $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are given by

$$\mathbb{Q}_1 = \begin{pmatrix} \mathbb{A}_1 & & & \mathbf{O} \\ & \mathbb{A}_1 & & \\ \mathbf{O} & & \ddots & \\ & & \mathbb{A}_1 & \\ & & & \mathbb{A}_1 \end{pmatrix}, \qquad \mathbb{Q}_2 = \left(\begin{array}{c|c} \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{B}_3 & \mathbb{B}_4 \end{array}\right), \tag{21}$$

where $\mathbb{Q}_2$ has the same identities as Q_N, i.e., $\mathbb{B}_j$ are all square-block matrices, $\mathbb{B}_3 = -\mathbb{B}_2^T$, $\mathbb{B}_4 = \mathbb{B}_1^T$, while $\mathbb{B}_1$ and $\mathbb{B}_2$ are expressible in the form

$$\mathbb{B}_1 = \begin{pmatrix} \mathbb{A}_2 & & & \mathbf{O} \\ & \mathbb{A}_2 & & \\ & & \ddots & \\ \mathbf{O} & & & \mathbb{A}_2 \end{pmatrix}, \qquad \mathbb{B}_2 = \begin{pmatrix} \mathbb{A}_3 & \mathbb{O} & \cdots & \mathbb{A}_{N-1} & \mathbb{A}_N \\ \mathbb{O} & \mathbb{A}_3 & \cdots & -\mathbb{A}_N & \mathbb{A}_{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\mathbb{A}_{N-1} & \mathbb{A}_N & \cdots & \mathbb{A}_3 & \mathbb{O} \\ -\mathbb{A}_N & -\mathbb{A}_{N-1} & \cdots & \mathbb{O} & \mathbb{A}_3 \end{pmatrix}, \tag{22}$$

$$\mathbb{A}_1 = \begin{pmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{pmatrix}, \qquad \mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{A}_j = \begin{pmatrix} u_{j+1} & 0 \\ 0 & u_{j+1} \end{pmatrix}, \quad (j = 2, 3, \ldots, N-1). \tag{23}$$

By a direct computation, one can easily check that system (7) can be derived from equation (16) by straightforward substitution of expressions (20)–(23). It should be noted that the above Lax pair expressed by the $2^N \times 2^N$ matrix representation is different from that form in [23]. By means of such a matrix expression, the Darboux transformation of system (7) can be further constructed to obtain a series of explicit solutions.

3. Darboux transformation

In this section, our concern is to construct the Darboux transformation of system (7). First of all, we consider the two-component mKdV equations including the self-focusing type case

$$\begin{aligned} &u_{1t} + 6\left(u_1^2 + u_2^2\right) u_{1x} + u_{1xxx} = 0, \\ &u_{2t} + 6\left(u_1^2 + u_2^2\right) u_{2x} + u_{2xxx} = 0, \end{aligned} \tag{24}$$

and the self-defocusing case

$$\begin{aligned} &u_{1t} - 6\left(u_1^2 + u_2^2\right) u_{1x} + u_{1xxx} = 0, \\ &u_{2t} - 6\left(u_1^2 + u_2^2\right) u_{2x} + u_{2xxx} = 0. \end{aligned} \tag{25}$$

We introduce the following gauge transformation

$$\widetilde{\Psi} = D\Psi = (\lambda I - S)\Psi, \tag{26}$$

where $\widetilde{\Psi}$ and Ψ are both four-dimensional vector eigenfunctions, I is a 4×4 identity matrix, D is called the Darboux matrix, while $\widetilde{\Psi}$ also satisfies the same linear eigenvalue problems (8) and (9) with P, V_0, V_1, V_2 and V_3 replaced by $\widetilde{P}$, $\widetilde{V}_0$, $\widetilde{V}_1$, $\widetilde{V}_2$ and $\widetilde{V}_3$, respectively,

$$\widetilde{\Psi}_x = \widetilde{U}\widetilde{\Psi} = (\lambda J + \widetilde{P})\widetilde{\Psi}, \tag{27}$$

$$\widetilde{\Psi}_t = \widetilde{V}\widetilde{\Psi} = (\lambda^3 \widetilde{V}_0 + \lambda^2 \widetilde{V}_1 + \lambda \widetilde{V}_2 + \widetilde{V}_3)\widetilde{\Psi}. \tag{28}$$

The compatibility condition $\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U}, \widetilde{V}] = O$ also gives rise to two-component mKdV equations (24) and (25). Thus, the Darboux matrix D is required to satisfy

$$D_x = \widetilde{U} D - DU, \tag{29}$$

$$D_t = \widetilde{V} D - DV, \tag{30}$$

from which we can directly compute out

$$\widetilde{V}_0 = V_0, \tag{31}$$

$$\widetilde{V}_1 = V_1 + [V_0, S], \tag{32}$$

$$\widetilde{V}_2 = V_2 + [V_1, S] + [V_0, S]S, \tag{33}$$

$$\widetilde{V}_3 = V_3 + [V_2, S] + [V_1, S]S + [V_0, S]S^2, \tag{34}$$

$$\widetilde{P} = P + [J, S], \tag{35}$$

$$S_x = [P, S] + [J, S]S, \tag{36}$$

$$S_t = [V_3, S] + [V_2, S]S, +[V_1, S]S^2 + [V_0, S]S^3. \tag{37}$$

The central task is to construct the matrix S based on the solutions of the linear eigenvalue problems (8) and (9). However, this problem is not easy to deal with, as the reduction $\widetilde{R} = \varepsilon \widetilde{Q}^T (\varepsilon = \pm 1)$ and constraints among potentials in $\widetilde{Q}$ and $\widetilde{R}$ should be consistent with the original reduction $R = \varepsilon Q^T (\varepsilon = \pm 1)$ and constraints in Q and R after the Darboux transformation. We take the matrix S of the form

$$S = H \Lambda H^{-1}, \tag{38}$$

with

$$H = (h_1, h_2, h_3, h_4), \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \tag{39}$$

where $h_j = (h_{1j}, h_{2j}, h_{3j}, h_{4j})^T (j = 1, 2, 3, 4)$ is the column solution of linear systems (8) and (9) with $\lambda = \lambda_j$, i.e.,

$$h_{j_x} = \lambda_j J h_j + P h_j, \tag{40}$$

$$h_{j_t} = \sum_{k=0}^{3} \lambda_j^{3-k} V_k h_j. \tag{41}$$

In order to hold the above reduction and constraints, λ_j and h_j have to satisfy certain relationship. The representations of the linear problems associated with the self-focusing and self-defocusing two-component mKdV equations described in the previous section allow us to construct the form of H.

Proposition 1. *The self-focusing two-component mKdV equations:*
Let

$$(h_1, h_2) = \begin{pmatrix} h_{11} & h_{21} \\ h_{21} & -h_{11} \\ h_{31} & h_{41} \\ h_{41} & -h_{31} \end{pmatrix}$$

is a vector solution of linear systems (8) and (9) with $\lambda = \lambda_1$, *then*

$$(h_3, h_4) = \begin{pmatrix} h_{31} & h_{41} \\ -h_{41} & h_{31} \\ -h_{11} & -h_{21} \\ h_{21} & -h_{11} \end{pmatrix}$$

is also a vector solution of linear systems (8) and (9) corresponding to $\lambda = -\lambda_1$.

Hence, the matrix S is given for the self-focusing two-component mKdV equations (24)

$$S = H\Lambda H^{-1}, \qquad H = \begin{pmatrix} h_{11} & h_{21} & h_{31} & h_{41} \\ h_{21} & -h_{11} & -h_{41} & h_{31} \\ h_{31} & h_{41} & -h_{11} & -h_{21} \\ h_{41} & -h_{31} & h_{21} & -h_{11} \end{pmatrix}, \qquad \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, -\lambda_1, -\lambda_1). \tag{42}$$

Proposition 2. *The self-defocusing two-component mKdV equations:*
Let

$$(h_1, h_2) = \begin{pmatrix} h_{11} & h_{21} \\ h_{21} & -h_{11} \\ h_{31} & h_{41} \\ h_{41} & -h_{31} \end{pmatrix}$$

is a vector solution of linear systems (8) and (9) with $\lambda = \lambda_1$, then

$$(h_3, h_4) = \begin{pmatrix} h_{31} & h_{41} \\ -h_{41} & h_{31} \\ h_{11} & h_{21} \\ -h_{21} & h_{11} \end{pmatrix}$$

is also a vector solution of linear systems (8) and (9) corresponding to $\lambda = -\lambda_1$.

For the self-defocusing case, S can be written as

$$S = H\Lambda H^{-1}, \qquad H = \begin{pmatrix} h_{11} & h_{21} & h_{31} & h_{41} \\ h_{21} & -h_{11} & -h_{41} & h_{31} \\ h_{31} & h_{41} & h_{11} & h_{21} \\ h_{41} & -h_{31} & -h_{21} & h_{11} \end{pmatrix}, \qquad \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, -\lambda_1, -\lambda_1). \tag{43}$$

Next we shall prove that the above matrix S satisfies expressions (32)–(34) together with expressions (36)–(37). By using equations (40) and (41), we take the derivatives of H with respect to x and t,

$$H_x = JH\Lambda + PH, \tag{44}$$

$$H_t = \sum_{k=0}^{3} V_j H\Lambda^{3-j}, \tag{45}$$

and further obtain

$$S_x = [H_x H^{-1}, S] = [JS + P, S], \tag{46}$$

$$S_t = [H_t H^{-1}, S] = \left[\sum_{k=0}^{3} V_k S^{3-k}, S\right]. \tag{47}$$

Obviously, the above expressions (46) and (47) exactly equal expressions (36) and (37), respectively. With the help of propositions 1 and 2, using the relation (35) together with expressions (42) and (43), one can find that expressions (32)–(34) are all automatically satisfied. To this stage, we have verified that the Darboux transformation (26) preserves the forms of the Lax pairs associated with the self-focusing and self-defocusing two-component mKdV equations, and keeps the reduction and constraints involved before. Thus, from expression (35), the relationship between the new and original potentials can be established.

4. *n*-times iteration of the Darboux transformation

The Darboux matrix $D = \lambda I - S$ discussed above is only one degree polynomial about λ. By applying the Darboux transformation successively, we can construct the n-times iteration of the Darboux transformation. Let us take $h_j (j = 1, 2, \ldots, 4n)$ as $4n$ fundamental solutions of linear systems (8) and (9) corresponding to eigenvalue parameters $\lambda = \lambda_j (j = 1, 2, \ldots, 4n)$. According to the knowledge about the Darboux transformation of higher degree [25, 26], the n-times iteration of the Darboux matrix is in the form of

$$D_n(x, t, \lambda) = \lambda^n I + \sum_{k=1}^{n} \Gamma_k \lambda^{n-k}, \tag{48}$$

which is the product of n Darboux transformation of degree one, namely,

$$D_n(x, t, \lambda) = (\lambda I - S_n)(\lambda I - S_{n-1}) \cdots (\lambda I - S_1). \tag{49}$$

Hence, from expressions (48) and (49), it is found that

$$\Gamma_1 = -(S_1 + S_2 + \cdots + S_n). \tag{50}$$

To this state, the n-times Darboux transformation for eigenfunction and potentials is presented as follows:

$$\Psi_n = \left(\lambda^n I + \sum_{k=1}^{n} \Gamma_k \lambda^{n-k} \right) \Psi, \tag{51}$$

$$P_n = P + [J, S_1 + S_2 + \cdots + S_n] = P - [J, \Gamma_1]. \tag{52}$$

The following step is to compute out the matrix Γ_1 in order to generate the new solutions from expression (52). Because $h_j (j = 1, 2, \ldots, 4n)$ is the column solution of linear systems (8) and (9) with $\lambda = \lambda_j$, it satisfies the linear algebraic equations $D_n(x, t, \lambda_j) h_j = 0$, i.e.,

$$\sum_{k=1}^{n} \Gamma_k \lambda_j^{n-k} h_j = -\lambda_j^n h_j, \qquad (j = 1, 2, \ldots, 4n), \tag{53}$$

which can be rewritten in a matrix form

$$(\Gamma_1, \Gamma_2, \ldots, \Gamma_n) W_n = -B, \tag{54}$$

with

$$B = \left(\lambda_1^n h_1, \lambda_2^n h_2, \ldots, \lambda_{4n}^n h_{4n}\right), \qquad W_n = \begin{pmatrix} \lambda_1^{n-1} h_1 & \lambda_2^{n-1} h_2 & \cdots & \lambda_{4n}^{n-1} h_{4n} \\ \lambda_1^{n-2} h_1 & \lambda_2^{n-2} h_2 & \cdots & \lambda_{4n}^{n-2} h_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ h_1 & h_2 & \cdots & h_{4n} \end{pmatrix}. \tag{55}$$

With the use of the Cramer's rule, Γ_1 can be solved from the linear algebraic equations (54),

$$(\Gamma_1)_{pq} = -\frac{\det M_q^{(p)}}{\det W_n}, \qquad (1 \leqslant p, q \leqslant 4), \tag{56}$$

where $M_q^{(p)}$ can be got by replacing the qth row of W_n with the pth row of B. Therefore, the potential formulae of n-times iterative Darboux transformation for the self-focusing and self-defocusing two-component mKdV equations are expressed as

$$u_1^{(n)} = u_1 + 2\mathrm{i}(\Gamma_1)_{13} = u_1 - 2\mathrm{i}\frac{\det M_3^{(1)}}{\det W_n}, \tag{57}$$

$$u_2^{(n)} = u_2 + 2\mathrm{i}(\Gamma_1)_{14} = u_2 - 2\mathrm{i}\frac{\det M_4^{(1)}}{\det W_n}. \tag{58}$$

5. Symbolic computation on the multi-soliton solutions

In what follows, applying the iterative algorithm of the Darboux transformation, we will perform the symbolic computation to construct the multi-soliton solutions of the self-focusing and self-defocusing two-component mKdV equations.

In order to generate the soliton solution, we take $u_1 = u_2 = 0$ as the seed solutions, and solve the linear eigenvalue problems (8) and (9) with $\lambda = \mathrm{i}\xi$ (ξ is an arbitrary real constant), then get the basic solution

$$h = \begin{pmatrix} h_{11} \\ h_{21} \\ h_{31} \\ h_{41} \end{pmatrix} = \begin{pmatrix} c_{11}\,\mathrm{e}^{\xi x - 4\xi^3 t} \\ c_{21}\,\mathrm{e}^{\xi x - 4\xi^3 t} \\ c_{31}\,\mathrm{e}^{-\xi x + 4\xi^3 t} \\ c_{41}\,\mathrm{e}^{-\xi x + 4\xi^3 t} \end{pmatrix}, \tag{59}$$

where c_{11}, c_{21}, c_{31} and c_{41} are all arbitrary real constants.

5.1. The self-focusing two-component mKdV equations

According to proposition 1 and the once-iterated potential formulae (57) and (58), we can derive the one-soliton solution of the self-focusing two-component mKdV equations (24), as below

$$u_1^{(1)} = \frac{2\xi(c_{11}c_{31} + c_{21}c_{41})}{\left(c_{11}^2 + c_{21}^2\right)\mathrm{e}^{\eta/2}}\,\mathrm{sech}\left[2(4\xi^3\mathrm{t} - \xi\mathrm{x}) + \frac{\eta}{2}\right], \tag{60}$$

$$u_2^{(1)} = \frac{2\xi(c_{11}c_{41} - c_{21}c_{31})}{\left(c_{11}^2 + c_{21}^2\right)\mathrm{e}^{\eta/2}}\,\mathrm{sech}\left[2(4\xi^3\mathrm{t} - \xi\mathrm{x}) + \frac{\eta}{2}\right], \tag{61}$$

where $\mathrm{e}^{\eta} = \left(c_{31}^2 + c_{41}^2\right)/\left(c_{11}^2 + c_{21}^2\right)$.

Adopting two sets of basic solutions (59), i.e., $h^{(1)} = \left(h_{11}^{(1)}, h_{21}^{(1)}, h_{31}^{(1)}, h_{41}^{(1)}\right)^T$ and $h^{(2)} = \left(h_{11}^{(2)}, h_{21}^{(2)}, h_{31}^{(2)}, h_{41}^{(2)}\right)^T$ corresponding to the eigenvalues λ_1 and λ_2, respectively, the two-soliton solutions can be generated from formulae (57) and (58)

$$u_1^{(2)} = -2\mathrm{i}\frac{\det M_3^{(1)}}{\det W_2}, \tag{62}$$

$$u_2^{(2)} = -2\mathrm{i}\frac{\det M_4^{(1)}}{\det W_2}, \tag{63}$$

with

$$M_3^{(1)} = \begin{pmatrix}
\lambda_1 h_{11}^{(1)} & \lambda_1 h_{21}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & \lambda_2 h_{11}^{(2)} & \lambda_2 h_{21}^{(2)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} \\
\lambda_1 h_{21}^{(1)} & -\lambda_1 h_{11}^{(1)} & -\lambda_1 h_{41}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_2 h_{21}^{(2)} & -\lambda_2 h_{11}^{(2)} & -\lambda_2 h_{41}^{(2)} & \lambda_2 h_{31}^{(2)} \\
\lambda_1^2 h_{11}^{(1)} & \lambda_1^2 h_{21}^{(1)} & \lambda_1^2 h_{31}^{(1)} & \lambda_1^2 h_{41}^{(1)} & \lambda_2^2 h_{11}^{(2)} & \lambda_2^2 h_{21}^{(2)} & \lambda_2^2 h_{31}^{(2)} & \lambda_2^2 h_{41}^{(2)} \\
\lambda_1 h_{41}^{(1)} & -\lambda_1 h_{31}^{(1)} & \lambda_1 h_{21}^{(1)} & -\lambda_1 h_{11}^{(1)} & \lambda_2 h_{41}^{(2)} & -\lambda_2 h_{31}^{(2)} & \lambda_2 h_{21}^{(2)} & -\lambda_2 h_{11}^{(2)} \\
h_{11}^{(1)} & h_{21}^{(1)} & h_{31}^{(1)} & h_{41}^{(1)} & h_{11}^{(2)} & h_{21}^{(2)} & h_{31}^{(2)} & h_{41}^{(2)} \\
h_{21}^{(1)} & -h_{11}^{(1)} & -h_{41}^{(1)} & h_{31}^{(1)} & h_{21}^{(2)} & -h_{11}^{(2)} & -h_{41}^{(2)} & h_{31}^{(2)} \\
h_{31}^{(1)} & h_{41}^{(1)} & -h_{11}^{(1)} & -h_{21}^{(1)} & h_{31}^{(2)} & h_{41}^{(2)} & -h_{11}^{(2)} & -h_{21}^{(2)} \\
h_{41}^{(1)} & -h_{31}^{(1)} & h_{21}^{(1)} & -h_{11}^{(1)} & h_{41}^{(2)} & -h_{31}^{(2)} & h_{21}^{(2)} & -h_{11}^{(2)}
\end{pmatrix}, \tag{64}$$

$$M_4^{(1)} = \begin{pmatrix} \lambda_1 h_{11}^{(1)} & \lambda_1 h_{21}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & \lambda_2 h_{11}^{(2)} & \lambda_2 h_{21}^{(2)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} \\ \lambda_1 h_{21}^{(1)} & -\lambda_1 h_{11}^{(1)} & -\lambda_1 h_{41}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_2 h_{21}^{(2)} & -\lambda_2 h_{11}^{(2)} & -\lambda_2 h_{41}^{(2)} & \lambda_2 h_{31}^{(2)} \\ \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & -\lambda_1 h_{11}^{(1)} & -\lambda_1 h_{21}^{(1)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} & -\lambda_2 h_{11}^{(2)} & -\lambda_2 h_{21}^{(2)} \\ \lambda_1^2 h_{11}^{(1)} & \lambda_1^2 h_{21}^{(1)} & \lambda_1^2 h_{31}^{(1)} & \lambda_1^2 h_{41}^{(1)} & \lambda_2^2 h_{11}^{(2)} & \lambda_2^2 h_{21}^{(2)} & \lambda_2^2 h_{31}^{(2)} & \lambda_2^2 h_{41}^{(2)} \\ h_{11}^{(1)} & h_{21}^{(1)} & h_{31}^{(1)} & h_{41}^{(1)} & h_{11}^{(2)} & h_{21}^{(2)} & h_{31}^{(2)} & h_{41}^{(2)} \\ h_{21}^{(1)} & -h_{11}^{(1)} & -h_{41}^{(1)} & h_{31}^{(1)} & h_{21}^{(2)} & -h_{11}^{(2)} & -h_{41}^{(2)} & h_{31}^{(2)} \\ h_{31}^{(1)} & h_{41}^{(1)} & -h_{11}^{(1)} & -h_{21}^{(1)} & h_{31}^{(2)} & h_{41}^{(2)} & -h_{11}^{(2)} & -h_{21}^{(2)} \\ h_{41}^{(1)} & -h_{31}^{(1)} & h_{21}^{(1)} & -h_{11}^{(1)} & h_{41}^{(2)} & -h_{31}^{(2)} & h_{21}^{(2)} & -h_{11}^{(2)} \end{pmatrix}, \tag{65}$$

$$W_2 = \begin{pmatrix} \lambda_1 h_{11}^{(1)} & \lambda_1 h_{21}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & \lambda_2 h_{11}^{(2)} & \lambda_2 h_{21}^{(2)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} \\ \lambda_1 h_{21}^{(1)} & -\lambda_1 h_{11}^{(1)} & -\lambda_1 h_{41}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_2 h_{21}^{(2)} & -\lambda_2 h_{11}^{(2)} & -\lambda_2 h_{41}^{(2)} & \lambda_2 h_{31}^{(2)} \\ \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & -\lambda_1 h_{11}^{(1)} & -\lambda_1 h_{21}^{(1)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} & -\lambda_2 h_{11}^{(2)} & -\lambda_2 h_{21}^{(2)} \\ \lambda_1 h_{41}^{(1)} & -\lambda_1 h_{31}^{(1)} & \lambda_1 h_{21}^{(1)} & -\lambda_1 h_{11}^{(1)} & \lambda_2 h_{41}^{(2)} & -\lambda_2 h_{31}^{(2)} & \lambda_2 h_{21}^{(2)} & -\lambda_2 h_{11}^{(2)} \\ h_{11}^{(1)} & h_{21}^{(1)} & h_{31}^{(1)} & h_{41}^{(1)} & h_{11}^{(2)} & h_{21}^{(2)} & h_{31}^{(2)} & h_{41}^{(2)} \\ h_{21}^{(1)} & -h_{11}^{(1)} & -h_{41}^{(1)} & h_{31}^{(1)} & h_{21}^{(2)} & -h_{11}^{(2)} & -h_{41}^{(2)} & h_{31}^{(2)} \\ h_{31}^{(1)} & h_{41}^{(1)} & -h_{11}^{(1)} & -h_{21}^{(1)} & h_{31}^{(2)} & h_{41}^{(2)} & -h_{11}^{(2)} & -h_{21}^{(2)} \\ h_{41}^{(1)} & -h_{31}^{(1)} & h_{21}^{(1)} & -h_{11}^{(1)} & h_{41}^{(2)} & -h_{31}^{(2)} & h_{21}^{(2)} & -h_{11}^{(2)} \end{pmatrix}. \tag{66}$$

5.2. The self-defocusing two-component mKdV equations

Similarly, following the procedure in section 5.1, we can derive the once-iterated exact analytical solutions of the self-defocusing two-component mKdV equations (25)

$$u_1^{(1)} = \frac{2\xi(c_{11}c_{31} + c_{21}c_{41})}{-\left(c_{11}^2 + c_{21}^2\right)e^{\eta/2}} \operatorname{csch}\left[-2(4\xi^3 \mathrm{t} - \xi \mathrm{x}) - \frac{\eta}{2}\right], \tag{67}$$

$$u_2^{(1)} = \frac{2\xi(c_{11}c_{41} - c_{21}c_{31})}{-\left(c_{11}^2 + c_{21}^2\right)e^{\eta/2}} \operatorname{csch}\left[-2(4\xi^3 \mathrm{t} - \xi \mathrm{x}) - \frac{\eta}{2}\right], \tag{68}$$

and twice-iterated solutions

$$u_1^{(2)} = -2\mathrm{i}\frac{\det M_3^{(1)}}{\det W_2}, \tag{69}$$

$$u_2^{(2)} = -2\mathrm{i}\frac{\det M_4^{(1)}}{\det W_2}, \tag{70}$$

with

$$M_3^{(1)} = \begin{pmatrix} \lambda_1 h_{11}^{(1)} & \lambda_1 h_{21}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & \lambda_2 h_{11}^{(2)} & \lambda_2 h_{21}^{(2)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} \\ \lambda_1 h_{21}^{(1)} & -\lambda_1 h_{11}^{(1)} & -\lambda_1 h_{41}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_2 h_{21}^{(2)} & -\lambda_2 h_{11}^{(2)} & -\lambda_2 h_{41}^{(2)} & \lambda_2 h_{31}^{(2)} \\ \lambda_1^2 h_{11}^{(1)} & \lambda_1^2 h_{21}^{(1)} & \lambda_1^2 h_{31}^{(1)} & \lambda_1^2 h_{41}^{(1)} & \lambda_2^2 h_{11}^{(2)} & \lambda_2^2 h_{21}^{(2)} & \lambda_2^2 h_{31}^{(2)} & \lambda_2^2 h_{41}^{(2)} \\ \lambda_1 h_{41}^{(1)} & -\lambda_1 h_{31}^{(1)} & -\lambda_1 h_{21}^{(1)} & \lambda_1 h_{11}^{(1)} & \lambda_2 h_{41}^{(2)} & -\lambda_2 h_{31}^{(2)} & -\lambda_2 h_{21}^{(2)} & \lambda_2 h_{11}^{(2)} \\ h_{11}^{(1)} & h_{21}^{(1)} & h_{31}^{(1)} & h_{41}^{(1)} & h_{11}^{(2)} & h_{21}^{(2)} & h_{31}^{(2)} & h_{41}^{(2)} \\ h_{21}^{(1)} & -h_{11}^{(1)} & -h_{41}^{(1)} & h_{31}^{(1)} & h_{21}^{(2)} & -h_{11}^{(2)} & -h_{41}^{(2)} & h_{31}^{(2)} \\ h_{31}^{(1)} & h_{41}^{(1)} & h_{11}^{(1)} & h_{21}^{(1)} & h_{31}^{(2)} & h_{41}^{(2)} & h_{11}^{(2)} & h_{21}^{(2)} \\ h_{41}^{(1)} & -h_{31}^{(1)} & -h_{21}^{(1)} & h_{11}^{(1)} & h_{41}^{(2)} & -h_{31}^{(2)} & -h_{21}^{(2)} & h_{11}^{(2)} \end{pmatrix}, \tag{71}$$

$$M_4^{(1)} = \begin{pmatrix} \lambda_1 h_{11}^{(1)} & \lambda_1 h_{21}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & \lambda_2 h_{11}^{(2)} & \lambda_2 h_{21}^{(2)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} \\ \lambda_1 h_{21}^{(1)} & -\lambda_1 h_{11}^{(1)} & -\lambda_1 h_{41}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_2 h_{21}^{(2)} & -\lambda_2 h_{11}^{(2)} & -\lambda_2 h_{41}^{(2)} & \lambda_2 h_{31}^{(2)} \\ \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & \lambda_1 h_{11}^{(1)} & \lambda_1 h_{21}^{(1)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} & \lambda_2 h_{11}^{(2)} & \lambda_2 h_{21}^{(2)} \\ \lambda_1^2 h_{11}^{(1)} & \lambda_1^2 h_{21}^{(1)} & \lambda_1^2 h_{31}^{(1)} & \lambda_1^2 h_{41}^{(1)} & \lambda_2^2 h_{11}^{(2)} & \lambda_2^2 h_{21}^{(2)} & \lambda_2^2 h_{31}^{(2)} & \lambda_2^2 h_{41}^{(2)} \\ h_{11}^{(1)} & h_{21}^{(1)} & h_{31}^{(1)} & h_{41}^{(1)} & h_{11}^{(2)} & h_{21}^{(2)} & h_{31}^{(2)} & h_{41}^{(2)} \\ h_{21}^{(1)} & -h_{11}^{(1)} & -h_{41}^{(1)} & h_{31}^{(1)} & h_{21}^{(2)} & -h_{11}^{(2)} & -h_{41}^{(2)} & h_{31}^{(2)} \\ h_{31}^{(1)} & h_{41}^{(1)} & h_{11}^{(1)} & h_{21}^{(1)} & h_{31}^{(2)} & h_{41}^{(2)} & h_{11}^{(2)} & h_{21}^{(2)} \\ h_{41}^{(1)} & -h_{31}^{(1)} & -h_{21}^{(1)} & h_{11}^{(1)} & h_{41}^{(2)} & -h_{31}^{(2)} & -h_{21}^{(2)} & h_{11}^{(2)} \end{pmatrix}, \tag{72}$$

$$W_2 = \begin{pmatrix} \lambda_1 h_{11}^{(1)} & \lambda_1 h_{21}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & \lambda_2 h_{11}^{(2)} & \lambda_2 h_{21}^{(2)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} \\ \lambda_1 h_{21}^{(1)} & -\lambda_1 h_{11}^{(1)} & -\lambda_1 h_{41}^{(1)} & \lambda_1 h_{31}^{(1)} & \lambda_2 h_{21}^{(2)} & -\lambda_2 h_{11}^{(2)} & -\lambda_2 h_{41}^{(2)} & \lambda_2 h_{31}^{(2)} \\ \lambda_1 h_{31}^{(1)} & \lambda_1 h_{41}^{(1)} & \lambda_1 h_{11}^{(1)} & \lambda_1 h_{21}^{(1)} & \lambda_2 h_{31}^{(2)} & \lambda_2 h_{41}^{(2)} & \lambda_2 h_{11}^{(2)} & \lambda_2 h_{21}^{(2)} \\ \lambda_1 h_{41}^{(1)} & -\lambda_1 h_{31}^{(1)} & -\lambda_1 h_{21}^{(1)} & \lambda_1 h_{11}^{(1)} & \lambda_2 h_{41}^{(2)} & -\lambda_2 h_{31}^{(2)} & -\lambda_2 h_{21}^{(2)} & \lambda_2 h_{11}^{(2)} \\ h_{11}^{(1)} & h_{21}^{(1)} & h_{31}^{(1)} & h_{41}^{(1)} & h_{11}^{(2)} & h_{21}^{(2)} & h_{31}^{(2)} & h_{41}^{(2)} \\ h_{21}^{(1)} & -h_{11}^{(1)} & -h_{41}^{(1)} & h_{31}^{(1)} & h_{21}^{(2)} & -h_{11}^{(2)} & -h_{41}^{(2)} & h_{31}^{(2)} \\ h_{31}^{(1)} & h_{41}^{(1)} & h_{11}^{(1)} & h_{21}^{(1)} & h_{31}^{(2)} & h_{41}^{(2)} & h_{11}^{(2)} & h_{21}^{(2)} \\ h_{41}^{(1)} & -h_{31}^{(1)} & -h_{21}^{(1)} & h_{11}^{(1)} & h_{41}^{(2)} & -h_{31}^{(2)} & -h_{21}^{(2)} & h_{11}^{(2)} \end{pmatrix}. \tag{73}$$

6. Conclusions

Recently, in soliton theory, many multi-component nonlinear partial differential equations emerge from fluid mechanics, nonlinear optics, Bose–Einstein condensates and field theories. Due to the tremendous significance of such dynamical multi-component models and of the associated interest in physics and mathematics, it is necessary to investigate the notion of integrability. The AKNS system has been proved to be an effective tool to find the Lax pairs and verify the integrability of many one-component soliton equations. In this paper, we have given a construction of Lax pair associated with the multi-component mKdV equations by extending the 2×2 matrix AKNS to $2^N \times 2^N$ situation. For the obtained Lax pair in terms of the block matrix, we have constructed the Darboux transformation of the self-focusing and self-defocusing two-component mKdV equations as illustrated examples, and presented the n-times iterative formula by applying the Darboux transformation successively. Performing the symbolic computation on the iterative algorithm of Darboux transformation, we have presented the multi-soliton solutions according to the zero background. Finally, for the higher component mKdV equations, the corresponding Darboux transformation can also be constructed through by following the procedure of two-component mKdV equations.

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References

[1] Barnett M P, Capitani J F, Von Zur Gathen J and Gerhard J 2004 *Int. J. Quantum Chem.* **100** 80
Tian B and Gao Y T 2005 *Phys. Plasmas* **12** 054701
Tian B and Gao Y T 2005 *Phys. Lett.* A **340** 243
Tian B and Gao Y T 2005 *Phys. Lett.* A **340** 449
Tian B and Gao Y T 2005 *Phys. Lett.* A **342** 228
Tian B and Gao Y T 2006 *Phys. Lett.* A **359** 241
Tian B and Gao Y T 2007 *Phys. Lett.* A **362** 283
Tian B, Gao Y T and Zhu H W 2007 *Phys. Lett.* A **366** 223
[2] Tian B and Gao Y T 2005 *Phys. Plasmas (Lett.)* **12** 070703
Tian B and Gao Y T 2005 *Eur. Phys. J.* D **33** 59
Gao Y T and Tian B 2007 *Phys. Lett.* A **361** 523
Gao Y T and Tian B 2007 *Europhys. Lett.* **77** 15001
[3] Zakharov V E 1991 *What Is Integrability?* (Berlin: Springer)
[4] Ablowitz M J and Clarkson P A 1992 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[5] Remoissenet M 1996 *Waves Called Solitons: Concepts and Experiments* (Berlin: Springer)
[6] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095
[7] Zakharov V E and Shabat A B 1972 *Sov. Phys.—JETP* **34** 62
[8] Wadati M 1972 *J. Phys. Soc. Japan* **32** 1681
[9] Ablowitz M J, Kaup D J, Newell A C and Segur H 1973 *Phys. Rev. Lett.* **31** 125
[10] Svinolupov S I 1992 *Commun. Math. Phys.* **143** 559
[11] Hirota R and Ohta Y 1991 *J. Phys. Soc. Japan* **60** 798
[12] Habibullin I T and Svinolupov S I 1995 *Physica* D **87** 134
[13] Yajima N and Oikawa M 1975 *Prog. Theor. Phys.* **54** 1576
[14] Zakharov V E and Kuznetsov E A 1986 *Physica* D **18** 455
[15] Athorne C and Fordy A P 1987 *J. Phys. A: Math. Gen.* **20** 1377
[16] Habibullin I T, Sokolov V V and Yamilov R I 1995 *Multi-component Integrable Systems and Nonassociative Structures in Nonlinear Physics: Theory and Experiment* (Singapore: World Scientific)
[17] Fordy A P and Kulish P P 1983 *Commun. Math. Phys.* **89** 427
[18] Kanna T, Lakshmanan M, Dinda P T and Akhmediev N 2006 *Phys. Rev.* E **73** 026604
[19] Zhang H Q, Meng X H, Xu T, Li L L and Tian B 2007 *Phys. Scr.* **75** 537
[20] Zhang H Q, Li J, Xu T, Zhang Y X, Hu W and Tian B 2007 *Phys. Scr.* **76** 452

[21] Wadati M and Tsuchida N 2006 *J. Phys. Soc. Japan* **75** 014301
[22] Zhang H Q, Xu T, Li J and Tian B 2008 *Phys. Rev.* E **77** 026605
[23] Tsuchida T and Wadati M 1998 *J. Phys. Soc. Japan* **67** 1175
[24] Iwao M and Hirota R 1997 *J. Phys. Soc. Japan* **66** 577
[25] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[26] Gu C H, Hu H S and Zhou Z X 2005 *Darboux Transformation in Soliton Theory and its Geometric Applications* (Shanghai: Shanghai Scientific and Technical Publishers)
[27] Doktorov E V and Leble S B 2007 *A Dressing Method in Mathematical Physics* (Netherlands: Springer)
[28] Cieśliński J L and Biernacki W 2005 *J. Phys. A: Math. Gen.* **38** 9491
[29] Xu T, Zhang H Q, Zhang Y X, Li J, Feng Q and Tian B 2008 *J. Math. Phys.* **49** 013501
[30] Neugebauer G and Meinel R 1984 *Phys. Lett.* A **100** 467
[31] Degasperis A and Lombardo S 2006 *Physica* D **214** 157
[32] Degasperis A and Lombardo S 2007 *J. Phys. A: Math. Theor.* **40** 961
[33] Park Q H and Shin H J 2001 *Physica* D **157** 1